

# Neutral edge modes in a superconductor – topological-insulator hybrid structure in a perpendicular magnetic field

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**Abstract** – We study the low-energy edge states of a superconductor – 3D topological-insulator hybrid structure (NS junction) in the presence of a perpendicular magnetic field. The hybridization of electron-like and hole-like Landau levels due to Andreev reflection gives rise to chiral edge states within each Landau level. We show that by changing the chemical potential of the superconductor, this junction can be placed in a regime where the sign of the effective charge of the edge state within the zeroth Landau level changes more than once resulting in *neutral* edge modes with a finite value of the guiding-center coordinate. The appearance of these neutral edge modes is related to the level repulsion between the zeroth and the first Landau levels in the spectra. We also find that these neutral edge modes come in *pairs*, one in the zeroth Landau level and its corresponding pair in the first. Unlike ordinary neutral *bogolon* excitations in superconductors, the neutral modes found by us have a finite speed and, thus, the potential to carry a heat current.

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**Introduction.** – The integer quantum Hall (QH) state [1] represented the first example of a many-body state whose nontrivial properties do not stem from a spontaneously-broken symmetry, but rather its topological character [2]. The precise quantisation of the Hall conductance in integer multiples of  $e^2/h$  was initially studied using a two-dimensional electron gas at a low temperature and in a strong perpendicular magnetic field, receiving a renewed attention with the discovery of graphene and its peculiar Dirac-like energy spectrum [3,4]. At the heart of transport in integer-QH systems – which based on their energy spectra could be expected to be insulating if the chemical potential lies in the gap between the neighbouring Landau levels (LLs) – is the existence of edge states. These LLs acquire dispersion as the guiding centers approach the physical edges of the sample (Hall bar), which gives rise to a current along the edges. The net current, resulting from a voltage (or chemical potential) drop in the direction perpendicular to the Hall-bar edges, is determined by the number of edge channels, *i.e.*, of LLs occupied in the bulk.

cally distinct from all previously known states of matter, led to a resurgence of interest in edge states. In this particular context, the peculiar feature of the ensuing topological insulators (TIs) [7] is that edge-state transport in these bulk-insulating materials takes place in a time-reversal-invariant fashion, *i.e.*, without an external magnetic field (see Refs. [8,9]). This is their principal difference from the conventional edge states in integer-QH systems.

Apart from integer-QH and TI systems, edge-state physics has in recent years been studied in heterostructures. The prime examples are the normal-superconductor (NS) junctions in a perpendicular magnetic field, whose N-part entails a system with a Dirac-like energy spectrum (either graphene or a TI). An additional physical ingredient in these systems – compared to the two aforementioned instances of edge states – is Andreev reflection [10] at the NS interface. An electron (charge  $-e$ ) incident from the N side is reflected as a positively-charged hole, while the missing charge of  $-2e$  enters the superconductor as an electron pair. These electron-hole conversion processes [11] and the ensuing Andreev edge states have been studied both theoretically [12] and experimentally [13] in

The advent of quantum spin-Hall states [5,6], topologi-

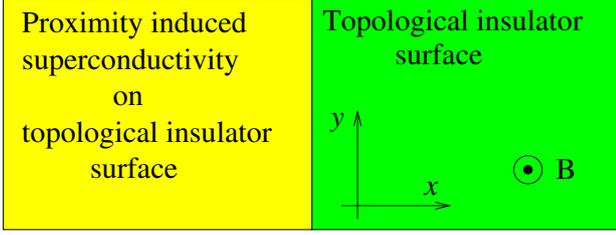


Fig. 1: (Colour online) Schematic of the NS heterostructure based upon a 3D topological insulator (TI). The left half represents the proximity-induced superconducting surface states of the TI, while the right one represents the normal surface states of the TI. A perpendicular magnetic field is applied in the right half.

graphene contacted with superconducting electrodes.

Quite recently, it was shown that a superconductor-TI junction in the presence of a perpendicular magnetic field can support a neutral chiral Majorana mode within the zeroth LL [14], which exists when the value of the guiding-center coordinate is *zero*. In contrast, in this article, we investigate edge states with a finite value of the guiding-center coordinate. We show that this junction supports additional *neutral* edge modes within the zeroth LL, which arise due to level repulsion between the zeroth and the first LLs in the spectrum. Using the Dirac-Bogoliubov-de Gennes equation [11, 12, 15], we study the spectrum of the dispersive edge states and show that this spectrum depends strongly on the chemical potential of the superconductor. We predict the existence of finite-energy, propagating neutral fermionic edge channels for a nonzero value of the guiding-center coordinate when the chemical potential of the superconductor is larger than a critical value  $\mu_c$ . These edge modes do not obey the self-conjugation property, which makes them fundamentally different from the Majorana modes studied in ref. [14].

**System and model.** – The system under investigation is shown in fig. 1. A conventional superconductor is deposited on the top surface of a 3D TI (depicted as  $x - y$  plane in fig. 1) inducing a finite superconducting pairing amplitude  $\Delta(\mathbf{r}) = e^{i\theta}\Delta_0$  in the left half ( $x < 0$ ) via proximity effect. A finite magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$  is present in the right half ( $x > 0$ ), where  $\mathbf{A}$  denotes the vector potential. The single-particle excitations in this N-S heterostructure can be described by the Dirac-Bogoliubov-de Gennes equation [11, 12, 14, 15]

$$\begin{pmatrix} H_D(\mathbf{r}) - \mu & \Delta(\mathbf{r})\sigma_0 \\ \Delta^*(\mathbf{r})\sigma_0 & \mu - \mathcal{T}H_D(\mathbf{r})\mathcal{T}^{-1} \end{pmatrix} \Psi(\mathbf{r}) = \varepsilon\Psi(\mathbf{r}). \quad (1)$$

Here  $\sigma_0 \equiv \mathbb{I}_{2 \times 2}$  denotes the two-dimensional identity matrix,  $\mathcal{T}$  the time-reversal operator, and  $H_D(\mathbf{r}) = v_F[\mathbf{p} + e\mathbf{A}(\mathbf{r})] \cdot \boldsymbol{\sigma}$  the massless-Dirac Hamiltonian for the TI surface states, with  $\boldsymbol{\sigma}$  being the vector of Pauli matrices that act in spin space. The position  $\mathbf{r} \equiv (x, y)$  and momentum  $\mathbf{p} \equiv -i\hbar(\partial_x, \partial_y)$  are restricted to the TI surface. In

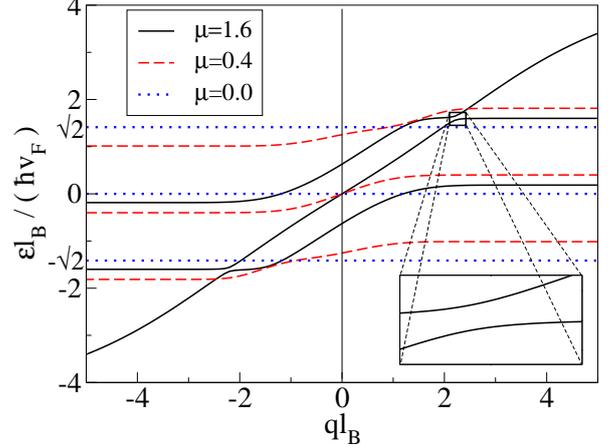


Fig. 2: (Colour online) Energy dispersion relation  $\varepsilon_n(q)$ , showing the zeroth and the first electron and hole-like ( $n = \pm 1$ ) Landau levels (LLs). Dotted (blue) lines are for  $\mu = 0$ , dashed (red) lines for  $\mu = 0.4\hbar v_F/l_B$ , and solid (black) lines for  $\mu = 1.6\hbar v_F/l_B$ . At  $q = 0$ , for all values of  $\mu$ , the zeroth LL has zero energy, the  $n = 1$  LL has positive energy, and the  $n = -1$  LL has negative energy. The value of the superconducting gap is  $\Delta_0 = 5\hbar v_F/l_B$ . The inset shows a zoom-in of the level repulsion between the zeroth and first LLs for  $\mu = 1.6\hbar v_F/l_B$ .

eq. (1) the excitation energy  $\varepsilon$  is measured relative to the chemical potential  $\mu$  of the superconductor, with the absolute zero of this energy set to be at the Dirac (*i.e.*, charge neutrality) point of the TI surface states. The wave function  $\Psi = (u_\uparrow, u_\downarrow, v_\downarrow, -v_\uparrow)^T$  in eq. (1) is a spinor in Dirac-Nambu space. Choosing the Landau gauge  $\mathbf{A} = Bx\hat{y}$  and demanding continuity of the wavefunction across the interface, we numerically obtain the energy-dispersion relation  $\varepsilon = \varepsilon(q)$ , where  $q$  is the guiding-center coordinate. The explicit form of the Dirac-Nambu spinor in the normal region is

$$\Psi(x, y) = e^{iqy} \times \begin{pmatrix} -iC_e e^{-(x+q)^2/2} (\mu + \varepsilon) H_{(\mu+\varepsilon)^2/2-1}(x+q) e^{i\theta/2} \\ C_e e^{-(x+q)^2/2} H_{(\mu+\varepsilon)^2/2}(x+q) e^{i\theta/2} \\ C_h e^{-(x-q)^2/2} H_{(\mu-\varepsilon)^2/2}(x-q) e^{-i\theta/2} \\ -iC_h e^{-(x-q)^2/2} (\mu - \varepsilon) H_{(\mu-\varepsilon)^2/2-1}(x-q) e^{-i\theta/2} \end{pmatrix}, \quad (2)$$

where  $H_\alpha(x)$  stands for the Hermite function [12, 14]. The coefficients  $C_e$  and  $C_h$  satisfy

$$\frac{C_e}{C_h} = \frac{-i\Delta_0(\mu - \varepsilon) H_{(\mu-\varepsilon)^2/2-1}(-q)}{\varepsilon H_{(\mu+\varepsilon)^2/2}(q) + (\mu + \varepsilon) H_{(\mu+\varepsilon)^2/2-1}(q) \sqrt{\Delta_0^2 - \varepsilon^2}}. \quad (3)$$

Here all lengths are measured in units of the magnetic length  $l_B \equiv \sqrt{\hbar/|eB|}$  and energies in units of  $\hbar v_F/l_B$ ; the dimensionless parameter representing the guiding center coordinate is  $ql_B$ .

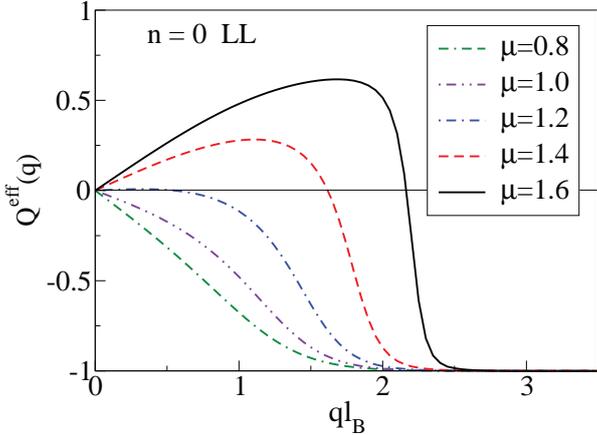


Fig. 3: (Colour online) Effective Nambu charge  $Q^{\text{eff}}(q)$  calculated using eq. (4) for different values of  $\mu$  (expressed in units of  $\hbar v_F/l_B$ ), for the zeroth LL.

**Results.** – Figure 2 shows the dispersion relation  $\varepsilon_n(q)$  of the single-particle excitations, where  $n$  denotes the LL index. Only the zeroth ( $n = 0$ ) and the first LLs ( $n = \pm 1$ ) are shown for  $\mu = 0$  (dotted lines),  $\mu = 0.4\hbar v_F/l_B$  (dashed lines), and  $\mu = 1.6\hbar v_F/l_B$  (solid lines). The dispersion relation has inversion symmetry  $\varepsilon_n(q) = -\varepsilon_{-n}(-q)$ . This energy spectrum contains a chiral Majorana mode within the zeroth Landau level around  $q = 0$  [14]. In the present article we focus on the level repulsion between different Landau levels at *finite*  $q$ . Figure 2 shows these level repulsions between the zeroth ( $n = 0$ ) and the first ( $n = 1$ ) LLs for  $\mu = 1.6\hbar v_F/l_B$  and a similar one between the zeroth ( $n = 0$ ) and the first hole-like ( $n = -1$ ) LLs. It should be noted that there is no such level repulsion for  $\mu = 0$  and  $\mu = 0.4\hbar v_F/l_B$ .

The effective Nambu charge

$$Q^{\text{eff}}(q) = \int_0^\infty dx [\Psi(x, y)]^\dagger \sigma_0 \otimes \tau_z \Psi(x, y) \quad (4)$$

of the Dirac-Andreev edge states described by  $\Psi(x, y)$  is generally not quantised [16–20] and can be calculated from the solution of eq. (1). Here  $\tau_z$  represents the usual Pauli  $z$  matrix that acts in the Nambu space. We observe that  $Q^{\text{eff}}(q)$  for the zeroth LL can change sign more than once. It should be noted that one sign change at  $q = 0$  is required as  $Q^{\text{eff}}(q)$  vanishes linearly for  $ql_B \ll 1$ , due to the presence of a chiral Majorana mode within the zeroth LL [14]. We find that this additional change of sign can be understood in terms of the level repulsion between the neighbouring LLs. Figure 3 shows the numerically computed  $Q^{\text{eff}}(q)$  for the zeroth LL for various different values of  $\mu$ . Only  $q > 0$  is shown in fig. 3 as the effective charge for the zeroth LL has inversion symmetry,  $Q^{\text{eff}}(q) = -Q^{\text{eff}}(-q)$ . For  $\mu = 0.8\hbar v_F/l_B$  and  $\mu = 1.0\hbar v_F/l_B$  we find that  $Q^{\text{eff}}(q)$  remains negative for all  $q > 0$ , while for  $\mu = 1.4\hbar v_F/l_B$

and  $\mu = 1.6\hbar v_F/l_B$  it changes sign, indicating the presence of a *neutral* fermionic mode at a finite value of  $q$ . The presence of this neutral fermionic mode has a direct correspondence with the level repulsion between the zeroth and the first LLs for  $\mu \gtrsim 1.2\hbar v_F/l_B$ .

For  $\mu = 0$ , the dispersionless zeroth LL has a negative effective charge over the entire domain  $q > 0$ , signifying its *electron-like* character. As  $\mu$  is increased, the zeroth LL changes character from *electron-like* to *hole-like* near the edge ( $0 < ql_B \lesssim 1$ ), while retaining its *electron-like* character deep inside the bulk ( $ql_B \gg 1$ ). This change necessitates the presence of a neutral fermionic mode for a finite value of the guiding-center coordinate  $q \neq 0$ . Although not directly measurable, the derivative of the effective Nambu charge

$$D^{\text{eff}}(q) = \frac{\partial Q^{\text{eff}}(q)}{\partial q} \quad (5)$$

is a convenient quantity to discuss the emergence of new neutral Andreev edge modes in our system. Figure 4 shows a plot of  $D^{\text{eff}}(q)$  for the zeroth LL and different values of  $\mu$ . We find that  $D^{\text{eff}}(q)$  remains a constant over most of its domain, vanishing for  $ql_B \gg 1$  (independent of  $\mu$ ). For  $ql_B \sim 1$ ,  $D^{\text{eff}}(q)$  shows a dip and finally approaches a constant value again for  $ql_B \ll 1$ . The sign of the constant to which  $D^{\text{eff}}(q)$  approaches for  $ql_B \ll 1$  depends upon the value of  $\mu$ . This neutral mode can be quantified further by calculating  $D^{\text{eff}}(q)$  at  $q = 0_+$ . Figure 5 shows a plot of  $D^{\text{eff}}(q)$  calculated at  $q = 0_+$  as a function of  $\mu$ . The critical value of the chemical potential  $\mu_c$  is indicated by the vanishing of  $D^{\text{eff}}(q)$  at  $q = 0_+$ . For  $\mu > \mu_c$  a neutral fermionic edge mode exists within the zeroth LL for a finite non-zero value of  $q$ . As mentioned before, this neutral fermionic mode is different from the chiral Majorana mode discussed in ref. [14], which exists within the zeroth LL at  $q = 0$  in that it does not satisfy the self-conjugation criterion, a defining property of the Majorana mode. The

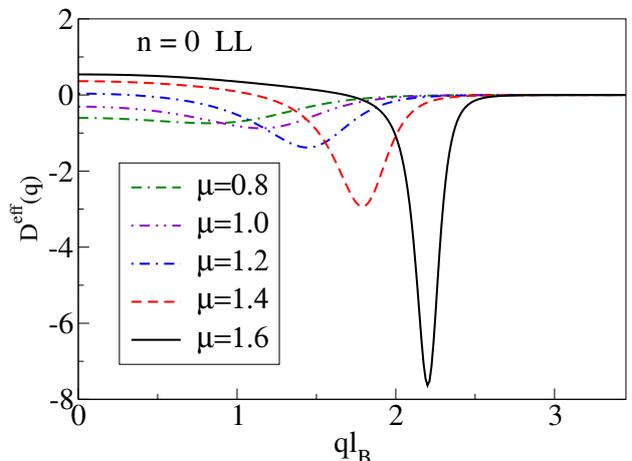


Fig. 4: (Colour online) Derivative of the effective Nambu charge  $D^{\text{eff}}(q) \equiv \partial Q^{\text{eff}}(q)/\partial q$  for the zeroth LL and different values of  $\mu$  (expressed in units of  $\hbar v_F/l_B$ ).

particle-hole conjugation operator is  $\Xi = \sigma_y \otimes \tau_y \mathcal{K}$ , where  $\mathcal{K}$  denotes complex conjugation,  $\sigma_y$  and  $\tau_y$  are the Pauli  $y$  matrices acting on spin and Nambu space respectively. Therefore, the particle-hole conjugate of this neutral mode at finite  $q$ , is another neutral mode at  $-q$ , also within the zeroth LL.

**Discussion.** – At the interface of the NS junction, the superconducting region hybridises the electron and the hole-like LLs resulting in dispersive edge modes with non-quantised effective charge within the superconducting gap. It should be noted that for  $\mu = 0$  the superconducting gap opens up at the Dirac point such that the *electron-like* branch of the superconductor is located strictly above the Dirac point (conduction band) while the *hole-like* branch is located strictly below the Dirac point (valence band). As  $\mu$  increases, part of the *hole-like* branch of the superconductor starts belonging to the conduction band. When  $\mu > \sqrt{2}\hbar v_F/l_B$  the entire  $n = -1$  LL belongs to the conduction band, putting the spectrum in the regime where level repulsion between the zeroth and the  $n = 1$  LL results in a neutral fermionic edge mode at finite positive energy and guiding-center coordinate. Similarly, level repulsion for the zeroth and the  $n = -1$  LL results in another neutral fermionic edge mode at finite negative energy and guiding-center coordinate.

The Andreev edge states discussed here are superpositions of electron and Nambu-hole Landau-level states. As  $\mu$  increases, neighbouring LLs get close in energy at a finite non-zero value of  $q$  (as shown in fig. 2) and interact. This interaction (level repulsion) between neighboring LLs involves states with opposite sign of charge. Level repulsion between these states hybridises them resulting in *neutral* Andreev edge modes at that particular value of  $q$ . This hybridisation always results in a pair of neutral edge modes, one in each LL participating in the level repulsion. To illustrate this further we show the numerically calculated effective Nambu charge  $Q^{\text{eff}}(q)$  for the first ( $n = 1$ ) LL in fig. 6 for different values of  $\mu$ . Comparing the vanishing of the effective charge for  $\mu = 1.4\hbar v_F/l_B$  and  $\mu = 1.6\hbar v_F/l_B$  in Figs. 3 and 6, we can see that the level repulsion between the zeroth and the first LLs results in neutral edge modes in each of them at the same value of the guiding-center coordinate.

Our results can be easily extended to a NS junction based on graphene, where each edge mode will have a twofold valley degeneracy assuming that the boundary conditions at the NS interface does not mix the two valleys [12]. As shown in ref. [21], the chiral Majorana mode existing within the zeroth LL at  $q = 0$  for TIs does not exist for graphene. However, we expect the neutral fermionic edge mode predicted here (for  $q \neq 0$ ) to survive in graphene. Recently, some progress has been made towards the experimental realisation of such heterostructures on graphene NS junctions [13].

The hybridisation of electron-like and hole-like quantum Hall edge channels was experimentally studied recently in

InAs/GaSb ambipolar quantum wells [22]. In these devices the hybridisation results in an energy gap over the whole sample width, which can be observed in transport measurements. Although similar in spirit, our system is quite different from the one studied in ref. [22]. In our NS junction the coupling between the electrons and the Nambu holes is provided by the superconducting interface via Andreev reflection. The superconducting region is therefore a necessary ingredient for the presence of the neutral edge modes discussed in this article. Moreover, in our case the level repulsion *does not* open an energy gap in the spectrum.

Strictly speaking, our calculation is only applicable to magnetic fields below the first critical field of the superconductor ( $B \lesssim B_{c1}$ ). For fields just above  $B_{c1}$  a small but finite number of vortices could change the properties of the edge modes discussed here if these vortices are located close to the edge (within distances of the order of the superconducting coherence length). The effect of these vortices can be reduced by pinning the vortices away from the edge. The order parameter induced in the left half ( $x < 0$ ) by the superconductor on top of the sample will also lead to a finite order parameter in the right half ( $x > 0$ ) which will decay exponentially away from the interface ( $x = 0$ ). However, the magnetic field applied to the right side will quench this weak induced order parameter since it is not supported by a genuine pairing interaction (see, *e.g.*, ref. [23] and references therein). As a result of this residual order parameter, or due to a suppression of the order parameter in the superconductor by the applied field, the actual NS interface may be slightly shifted from  $x = 0$ . However, our conclusions do not depend on the exact geometric location of the electronic interface.

One way to investigate the effective charge of the edge modes discussed here would be to generate an edge magnetoplasmon wave packet by using a voltage pulse. The shape of this wave packet would crucially depend on the effective charge of the fundamental density excitations. Fur-

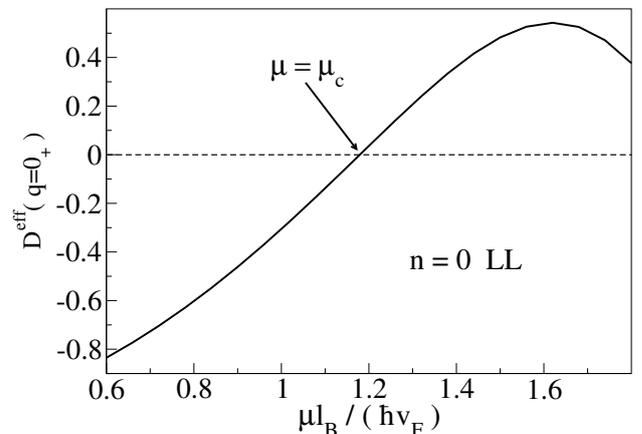


Fig. 5: Derivative of the the effective Nambu charge  $D^{\text{eff}}(q = 0_+)$  as a function of the chemical potential  $\mu$  for the zeroth LL.

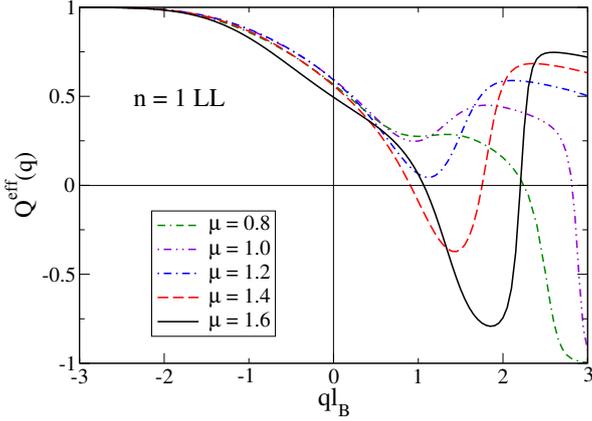


Fig. 6: (Colour online) Effective Nambu charge  $Q^{\text{eff}}(q)$  calculated using eq. (4) for different values of  $\mu$  (expressed in units of  $\hbar v_F/l_B$ ), for the first ( $n = 1$ ) LL.

thermore, even though charge will not be transported by the neutral mode, heat will. This is because the group velocity  $v_g = \partial \varepsilon_n(q)/\partial(\hbar q)$  does not vanish at the value of  $q$  at which  $Q^{\text{eff}} = 0$ . (For example, the neutral mode occurring in the  $n = 0$  Landau level at  $ql_B \approx 2.2$  has  $v_g = 0.53 v_F$ , which corresponds to 79% of the chiral Majorana mode's velocity.) Thus, by comparing heat and charge transport experiments, it should be possible to establish the vanishing charge of these excitations.

**Conclusion.** – We have investigated the spectrum of dispersive edge states in a superconductor – topological-insulator junction in the presence of a perpendicular magnetic field. The spectrum can change dramatically if the chemical potential  $\mu$  of the superconductor is raised above some critical value  $\mu_c$ . We show that the zeroth Landau level supports additional *neutral* edge modes at finite values of the guiding-center coordinate  $q$  when  $\mu > \mu_c$ , along with a neutral Majorana mode at  $q = 0$ . By investigating the energy spectra for  $\mu > \mu_c$  we found that these neutral modes are related to level repulsion between the zeroth LL and the first ( $n = \pm 1$ ) LLs. We also found that there are neutral modes at finite positive and negative values of the guiding-center coordinate in  $n = 1$  and  $n = -1$  LLs respectively, corresponding to the same values of  $q$  at which there are neutral edge modes in the zeroth LL. By calculating the effective Nambu charge and its derivative we have determined the critical chemical potential  $\mu_c$  above which these neutral edge modes exist. They arise due to the hybridisation of electron-like and hole-like LLs involved in the level repulsion. An experimental realisation of the heterostructures that we have investigated will provide further insight into the nature of exotic Andreev edge state in the presence of a perpendicular magnetic field.

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