1. Chain rule for Stratonovich SDE

The chain rule for the Itô calculus, i.e., Itôs formula [for t-independent $f(\hat{x}(t),t) = f(\hat{x}(t))$]

$$\mathrm{d}f(\hat{x}(t)) = \left\{a_{\mathrm{I}}(\hat{x}(t),t)f'(\hat{x}(t)) + \frac{1}{2}[b_{\mathrm{I}}(\hat{x}(t),t)]^2 f''(\hat{x}(t))\right\}\mathrm{d}t + b_{\mathrm{I}}(\hat{x}(t),t)\, f'(\hat{x}(t))\,\mathrm{d}\hat{W}(t) \quad (1)$$

contains a second order derivative with respect to x. Here, $\hat{x}(t)$ has to obey the Itô SDE

$$\mathrm{d}\hat{x}(t) = a_{\mathrm{I}}(\hat{x}(t),t)\,\mathrm{d}t + b_{\mathrm{I}}(\hat{x}(t),t)\,\mathrm{d}\hat{W}(t) \quad (2)$$

We will now demonstrate that the chain rule for the Stratonovich calculus is more straightforwardly given by the common chain rule of differential algebra,

$$\mathrm{d}f(\hat{x}(t)) = \left\{a_{\mathrm{S}}(\hat{x}(t),t)\mathrm{d}t + b_{\mathrm{S}}(\hat{x}(t),t)\circ\mathrm{d}\hat{W}(t)\right\} f'(\hat{x}(t)) = \mathrm{d}\hat{x}(t)\, f'(\hat{x}(t))\,, \quad (3)$$

where $\hat{x}(t)$ has to fulfill the Stranonovich SDE implicitely defined by the second equality sign, i.e., Eq. (4) below.

1.1. Transformation

Show for the one-dimensional case that the Stratonovich SDE

$$\mathrm{d}\hat{x}(t) = a_{\mathrm{S}}(\hat{x}(t),t)\,\mathrm{d}t + b_{\mathrm{S}}(\hat{x}(t),t)\circ\mathrm{d}\hat{W}(t) \quad (4)$$

is equivalent to the Itô SDE

$$\mathrm{d}\hat{x}(t) = \left[a_{\mathrm{S}}(\hat{x}(t),t) + v^{(\frac{1}{2})}(\hat{x}(t),t)\right]\mathrm{d}t + b_{\mathrm{S}}(\hat{x}(t),t)\,\mathrm{d}\hat{W}(t)\,, \quad (5)$$

where the noise-induced drift is given by

$$v^{(\frac{1}{2})}(x,t) = \frac{1}{2}\, b_{\mathrm{S}}(x,t)\,\frac{\partial}{\partial x}\, b_{\mathrm{S}}(x,t)\,. \quad (6)$$

1.2. Chain rule

Use Itô's formula (1) and convert the Itô SDE for the function $y = f(x)$ back to the Stratonovich SDE to derive expression (3).

2. Product rule (partial integration) for Itô processes

Let $\hat{x}(t)$ and $\hat{y}(t)$ be two Itô processes, i.e. solutions of the Itô SDEs

$$\begin{aligned}
\mathrm{d}\hat{x} &= a_x(\hat{x},t)\,\mathrm{d}t + b_x(\hat{x},t)\,\mathrm{d}\hat{W}(t)\\
\mathrm{d}\hat{y} &= a_y(\hat{y},t)\,\mathrm{d}t + b_y(\hat{y},t)\,\mathrm{d}\hat{W}(t)
\end{aligned}$$

with identical realizations of the Wiener process $\hat{W}(t)$. Derive using $\mathrm{d}\big[\hat{x}\hat{y}\big] = \hat{x}\,\mathrm{d}\hat{y} + \hat{y}\,\mathrm{d}\hat{x} + \mathrm{d}\hat{x}\,\mathrm{d}\hat{y}$ that the product $\hat{x}(t)\hat{y}(t)$ obeys the Itô process

$$\mathrm{d}\big[\hat{x}\hat{y}\big] = \big[\hat{x}\,a_y(\hat{y},t) + a_x(\hat{x},t)\,\hat{y} + b_x(\hat{x},t)b_y(\hat{y},t)\big]\mathrm{d}t + \big[\hat{x}\,b_y(\hat{y},t) + b_x(\hat{x}(t),t)\,\hat{y}\big]\mathrm{d}\hat{W}(t)$$

Verify that this is consistent with the multidimensional Itô formula given in the lecture.

Now assume that the two Wiener processes for $\hat{x}(t)$ and $\hat{y}(t)$ are independent. Which result for $\mathrm{d}[\hat{x}\hat{y}]$ do you then obtain?

3. Harmonic oscillator

The equation of motion for a classical harmonic oscillator is given by

$$m\ddot{\hat{x}}(t) + \eta\dot{\hat{x}}(t) + k\hat{x}(t) = \sqrt{2\eta k_B T}\;\hat{\xi}(t),$$

where $\hat{\xi}(t)$ describes Gaussian white noise with mean value $\langle\hat{\xi}(t)\rangle = 0$ and $\langle\hat{\xi}(t)\hat{\xi}(t')\rangle = \delta(t-t')$.

If we restrict ourselves to the stationary case, it is advantageous to consider the Fourier transform $\hat{x}(\omega) = \int \mathrm{d}t\;\mathrm{e}^{\mathrm{i}\omega t}\;\hat{x}(t)$ of the process. Derive an equation for $\hat{x}(\omega)$.

3.1. Mean value and autocorrelation function

Calculate the mean value $\langle\hat{x}(t)\rangle$ and the autocorrelation function $\langle\hat{x}(t)\hat{x}(t')\rangle$ for the stationary process for the underdamped case $\gamma = \frac{\eta}{m} < 2\,\omega_0$ where $\omega_0^2 = \frac{k}{m}$.