1. Discrete-time random walk

A drunkard moves along a line by making each second a step to the right or to the left with equal probability. Thus his possible positions are the integers $-\infty < n < \infty$. We assume that initially $n = 0$ and want to determine the probability $p_n(r)$ after r steps.

1.1. Combinatorial derivation

Derive the probability $p_n(r)$ by combinatorial arguments.
Hint: How many different path lead to the position n after r steps?

1.2. Solve by addition of variables

Each step $j = 1, 2, \ldots, r$ corresponds to a stochastic variable $\hat{x}_j$ taking on the values 1 and -1 with equal probability 1/2. The position after r steps is then given by

$$\hat{n}_r = \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_r.$$

The steps and thus the variables $\hat{x}_j$ are mutually independent.

1.2.1. Average

Derive the average $\langle \hat{n}_r \rangle$ and the variance $\langle \hat{n}_r^2 \rangle$. Discuss the r-dependence of these variables and compare to that of a deterministic (sober) movement.

1.2.2. Characteristic Function

Calculate the characteristic function $G_{\hat{n}_r}(k)$. Determine the probability $p_n(r)$ from this expression. What is the advantage of this method compared to the combinatorial derivation?

Hint:

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k$$

1.3. Asymmetric random walk

Now assume that a step to the left has the probability q and to the right $1-q$. Find $p_n(r)$ for this case.

2. Compound distribution

Let $\hat{x}_j$ be an infinite set of independent stochastic variables with identical distributions $P_{\hat{x}}(x)$ and characteristic function $G_{\hat{x}}(k) = \int\limits_{-\infty}^{\infty} P_{\hat{x}}(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$. Let $\hat{r}$ be a random non-negative integer with distribution p_r and probability generating function $f(z) = \langle z^{\hat{r}} \rangle = \sum\limits_{r=0}^{\infty} p_r z^r$. Then the sum

$$\hat{y} = \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\hat{r}}$$

is a random variable, where we set $\hat{y} = 0$ for $\hat{r} = 0$. Show that its characteristic function $G_{\hat{y}}(k)$ fulfills

$$G_{\hat{y}}(k) = f(G_{\hat{x}}(k)).$$

The distribution of $\hat{y}$ is the so-called *compound distribution*.

2.1. Example

Assume that $\hat{x}_i$ are Bernoulli variables, i.e. $P(\hat{x}_i = 1) = p = 1-q$ and $P(\hat{x}_i = -1) = q$ like in the asymetric random walk in 1.3, and $\hat{r}$ obeys Poisson statistics. How does the compound distribution $P_{\hat{y}}$ then look like?

3. Branching Process

We now consider a so-called *branching process* representing a kind of chain reaction occuring in successive generations. Initially, for the zeroth generation, we start with a single event.

This event causes in the first generation $\hat{r}$ further events, where $\hat{r}$ is a random non-negative integer with distribution p_r. If $\hat{r} = 0$, the process stops. Otherwise, every direct descendant causes in the next (second) generation further events according to the same distribution p_r and so on. The events of each generation act independently of each other.

Whereas an obvious example for such a process are nuclear chain reactions, originally this process has been invented by F. Galton to study the survival of family names (hence the name *Galton-Watson process*). Another example is the cascading failure of components like power lines which is induced by an overloading occuring after the initial failure of a line.

3.1. Recursion relation

Let $\hat{y}_n$ be the number of events, i.e., the size of the n-th generation with generating function $f_n(z)$. In particular, $y_0 = 1$ in the 0th generation. Furthermore, $f_1(z) = f(z)$, where $f(z) = \langle z^{\hat{r}} \rangle$ is the generating function corresponding to the variable $\hat{r}$. Derive from the description of the branching process, the recursive relation

$$f_n(z) = f(f_{n-1}(z))\,. \tag{1}$$

3.2. Probability for a finite cascade

An important question is whether the branching process continues forever or stops after a finite number of generations, i.e., whether $\hat{y}_n = 0$ for an arbitrary n.

3.2.1. Recursion formula

How can the probability $q_n := P(\hat{y}_n = 0)$ that the process stops at or before the n-th generation be calculated from the generating function $f_n(z)$? Derive from Eq. (1) a recursion formula for the q_n. What is the initial condition q_1?

3.2.2. Termination of branching process

Excluding the extreme cases $p_0 = 0$ and $p_0 = 1$ derive from the form of the generating function $f(z)$ that the q_n approach a limiting value $q \leq 1$ satisfying

$$q = f(q)\,. \tag{2}$$

Show geometrically that equation (2) has the unique solution $q = 1$ if and only if the first derivative $f'(1)$ of the characteristic function satisfies $f'(1) \leq 1$.
Hint: The characteristic function is convex. Why?

3.2.3. Mean size of generations

Show that the expected size of the n-th generation is given by

$$\langle \hat{y}_n \rangle = f_n'(1).$$

Derive a chain rule for $\langle \hat{y}_n \rangle$ and show that

$$\langle \hat{y}_n \rangle = \mu^n,$$

where μ is the mean value of $\hat{r}$. Why is it intuitively clear that the branching process dies out for $f'(1) < 1$?