1. Process with independent increments

By a process with independent increments we mean a familiy of random variables $\hat{\mathbf{x}}(t)$ depending on the continuous time parameter t such that the increments

$$\hat{\mathbf{x}}(t_{k+1}) - \hat{\mathbf{x}}(t_k)$$

are mutually independent for any finite set of $t_1 < t_2 < \ldots < t_n$.

In the following, we consider a time-homogenous process with independent increments using the forward equation

$$\frac{\partial}{\partial t} p_{1|1}(\mathbf{x}, t \mid \mathbf{x}_0, t_0) = \int \mathrm{d}^r x \, \Gamma(\mathbf{x}, \mathbf{x}'; t) \, p_{1|1}(\mathbf{x}', t | \mathbf{x}_0, t_0) \,,$$

where the master operator can be written in the form

$$\Gamma(\mathbf{x}, \mathbf{x}', t) = \Gamma(\mathbf{x} - \mathbf{x}') = \gamma \left[\rho(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}') \right] \,,$$

with γ being the jump rate and ρ the probability density function of the jump widths.

1.1. Conditional probability

Show that the conditional probability $p_{1|1}(\mathbf{x}, t \mid \mathbf{x}_0, t_0)$ $(t_0 \leq t)$ consequently also depends only on the spatial difference $\mathbf{x} - \mathbf{x}_0$ and time dependence $t - t_0$.

1.2. Solution for transition probability

Show that in terms of the Fourier transforms with respect to $\mathbf{x}$,

$$\begin{aligned} p_{1|1}(\mathbf{k}, t) &:= \int \mathrm{d}^r x \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \, p_{1|1}(\mathbf{x}, t \mid \mathbf{0}, 0) \quad \text{and} \\ \rho(\mathbf{k}) &:= \int \mathrm{d}^r x \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \, \rho(\mathbf{x}) \,, \end{aligned}$$

the solution of the forward equation is given by

$$p_{1|1}(\mathbf{k}, t) = \exp\left\{ \gamma \left[\rho(\mathbf{k}) - 1 \right] t \right\} \,.$$

Why does this imply that for asymptotic times $t \to \infty$, the process becomes Gaussian (provided that $\rho(\mathbf{k})$ is sufficiently smooth around $\mathbf{k} = \mathbf{0}$)?

1.3. Interpretation as compound Poisson distribution

Show that the back transformation yields

$$p_{1|1}(\mathbf{x},t \mid \mathbf{0},0) \;=\; \sum_{n=0}^{\infty} p_n(t) \Big\langle \delta\big(\mathbf{x} - \sum_{i=1}^{n} \hat{\mathbf{x}}_i\big) \Big\rangle_\rho$$

where $p_n(t) = \mathrm{e}^{-\gamma t}\,(\gamma t)^n/n!$, the jumps $\hat{\mathbf{x}}_i$ are independent, identically distributed according to the distribution $\rho(\mathbf{x})$ and $\langle \ldots \rangle_\rho$ denotes the corresponding average. Interpret the result in connection with a distribution from problem set 3.

2. Non-Gaussian white noise

Now we consider a process $\hat{\xi}(t)$ which fulfills

$$\begin{aligned} \langle \hat{\xi}(t) \rangle &= \Gamma_1\,, \\ \langle\!\langle \hat{\xi}(t)\hat{\xi}(t') \rangle\!\rangle &= \Gamma_2\, \delta(t-t')\,, \end{aligned}$$

but is not necessarily Gaussian. Thus, the higher order cumulants do not need to vanish, but we assume that they are δ-correlated according to

$$\langle\!\langle \hat{\xi}(t_1)\hat{\xi}(t_2)\ldots\hat{\xi}(t_m) \rangle\!\rangle \;=\; \Gamma_m\, \delta(t_1-t_2)\,\delta(t_1-t_3)\ldots\delta(t_1-t_m)$$

for $m \geq 2$. The Γ_m are arbitrary time-independent constants.

2.1. Characteristic functional

Calculate the generating functional $\Big\langle \exp\Big[\mathrm{i} \int \mathrm{d}t\, k(t)\hat{\xi}(t)\Big]\Big\rangle$ and its logarithm.

2.2. Characteristic function of integrated noise process

Mathematically better behaved is the integrated process

$$\hat{W}(t) = \int_0^t \hat{\xi}(t')\mathrm{d}t'.$$

Convince yourself that the $\hat{W}(t)$ is a process with independent increments.

Derive the characteristic function for the increments of the integrated process from the characteristic functional of the previous section by using $k(t) = k\;\theta(t-t_1)\theta(t-t_2)$.

2.2.1. Compound Poisson process

Discuss your results in terms of the compound Poisson process derived in exercise 1. In particular, derive the relation of the Γ_n with the jump rate and the moments of the jump width distribution.

Note that, vice versa, a white noise process can be constructed by taking any process with independent increments and differentiating it.

2.3. Example: Poisson process

Show that the Poisson process

$$p_{1|1}(n_2, t_2 \mid n_1, t_1) = \frac{(t_2 - t_1)^{n_2 - n_1}}{(n_2 - n_1)!} \mathrm{e}^{-(t_2 - t_1)}$$

with $p_1(n, 0) = \delta_{n,0}$ has independent increments. Show that this process provides white noise with $\Gamma_m = 1$ for all m.