1. Fokker-Planck equation

Assume $t = t_0 + \Delta t$ in the Fokker-Planck equation for $p_{1|1}(x, t|x_0, t_0)$

$$\frac{\partial p_{1|1}(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x}\left[A(x)p_{1|1}(x, t|x_0, t_0)\right] + \frac{\partial^2}{\partial x^2}\left[B(x)p_{1|1}(x, t|x_0, t_0)\right]. \tag{1}$$

Compute for small Δt the moments of $\Delta x = x - x_0$. Show that for $\Delta t \to 0$ the conditional expectation values obey

$$\begin{aligned}
\frac{\langle \Delta x | x_0, t_0 \rangle}{\Delta t} &= A(x_0), \\
\frac{\langle (\Delta x)^2 | x_0, t_0 \rangle}{\Delta t} &= 2B(x_0), \\
\frac{\langle (\Delta x)^\nu | x_0, t_0 \rangle}{\Delta t} &= 0 \quad \text{for} \quad \nu \geq 3.
\end{aligned}$$

Interpret this result. What is the descriptive meaning of the two coefficients A and B?

2. Periodic boundary conditions

We assume that a process takes place on an interval $[a, b]$ in which the two end points are identified with each other, e.g. diffusion in presence of an external force (drift) on a circle. A promiment example is that of a ratchet and a pawl discussed in Vol. I of the Feynman Lectures on Physics, where x can be identified with the angle of rotation of the ratchet.

Then the two boundary conditions have to be fulfilled

$$\lim_{x \to b^-} p(x, t) = \lim_{x \to a^+} p(x, t) \tag{2}$$

$$\lim_{x \to b^-} J(x, t) = \lim_{x \to a^+} J(x, t) \tag{3}$$

where the probability current $J(x, t)$ is defined by

$$J(x, t) = A(x, t)p_{1|1}(x, t|x_0, t_0) - \frac{\partial}{\partial x}\left[B(x, t)p_{1|1}(x, t|x_0, t_0)\right]. \tag{4}$$

2.1. Homogeneous Process

Now we assume a homeogeneous process such that the drift and diffusion constant are time independent. Rewrite the Fokker-Planck equation, Eq. (1), in terms of the current. Which property does a current thus have to fulfill in a stationary situation?

2.2. Periodic Boundary current

In contrast to the situation with, e.g., reflecting boundary conditions, where one has to require $J = 0$, the current does not need to vanish in the presence of periodic boundary conditions. It is determined by the normalization condition for the probability density and the periodic boundary conditions (2, 3). Integrate Eq. (4) to find

$$J = p_s(a) \left[B(b)/\Psi(b) - B(a)/\Psi(a) \right] / \left[\int_a^b dx'/\Psi(x') \right] ,$$

where we have introduced

$$\Psi(x) = \exp \left[\int_a^x dx' A(x')/B(x') \right] .$$

Derive an expression for the stationary probability $p_s(x)$ from this expression.

2.3. Condition for vanishing current

Assume now that $B(x) = D$ with the diffusion constant D. The sign of the current depends on the sign of the prefactor $[1/\Psi(b) - 1/\Psi(a)]$ as derived in the previous section. What is the condition that the current J is zero, positive or negative?

Give a simple sufficient condition on the potential of $A(x)$ which guarantees that $J = 0$. Interprete this hopefully surprising result—think about an asymmetric, “ratchet”, potential—in terms of the second law of thermodynamics.

In order to generate a ratchet current even in the absence of a net force, i.e., to obtain work from fluctuations, one has to consider non-equilibrium situations, e.g. by introducing a time-periodic external force or non-equilibrium noise. A plethora of such Brownian motor models has been discussed in the literature, see e.g. P. Reimann, *Brownian motors: noisy transport far from equilibrium*, Phys. Rep. **361**, 57 (2002) for a comprehensive review.