1. Problem: Some applications of Itô's lemma (10 points)

This exercise demonstrates some applications of Itô's lemma

$$\mathrm{d}f[\hat{W}(t), t] = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)\mathrm{d}t + \frac{\partial f}{\partial W}\,\mathrm{d}\hat{W}(t)\,. \tag{1}$$

1.1. Powers of the Wiener process

Derive using Eq. (1) the following expression for the Itô integral of the n-th ($n = 1, 2, \dots$) power of the Wiener process:

$$\int_{t_0}^{t} \left[\,\hat{W}(t')\right]^n \mathrm{d}\hat{W}(t') = \frac{1}{n+1}\left\{\left[\hat{W}(t)\right]^{n+1} - \left[\hat{W}(t_0)\right]^{n+1}\right\} - \frac{1}{2}\,n\int_{t_0}^{t}\left[\hat{W}(t)\right]^{n-1}\mathrm{d}t\,. \tag{2}$$

Evaluate this result in the special case $n = 1$ and check that the mean value $\left\langle \int_{t_0}^{t} \hat{W}(t')\,\mathrm{d}\hat{W}(t')\right\rangle$ vanishes. Why does this expectation values also have to vanish for the n-th power of the Wiener process?

Derive from the expression obtained for the integral (2) the following recursive relation for the expectation value of powers of the Wiener process:

$$\left\langle\left[\hat{W}(t)\right]^{n+1}\right\rangle = \frac{1}{2}n(n+1)\int_0^t \left\langle\left[\hat{W}(t)\right]^{n-1}\right\rangle \mathrm{d}t$$

Derive an explicit expression for $\langle[\hat{W}(t)]^{n+1}\rangle$ by using the first two moments $\langle\hat{W}(t)\rangle = 0$ and $\langle[\hat{W}(t)]^2\rangle = t$. How could one have obtained this directly from the Gaussian property of the Wiener process?

1.2. Exponential of the Wiener process

Obtain using Itô's lemma the following relation for the Itô integral of the exponential of a Wiener process:

$$\int_{t_0}^{t} \mathrm{e}^{\hat{W}(t')}\,\mathrm{d}\hat{W}(t') = e^{\hat{W}(t)} - e^{\hat{W}(t_0)} - \frac{1}{2}\int_{t_0}^{t} e^{\hat{W}(t')}\,\mathrm{d}t'\,.$$

Derive from this expression a differential equation for the expectation value $\langle \exp[\hat{W}(t)] \rangle$ and solve this equation using an appropriate initial condition at time zero.

2. Problem: Generalized stochastic integrals (no points)

The stochastic integral $\int_{t_0}^{t} \hat{g}(t') \mathrm{d}\hat{W}(t')$ is defined as a kind of Riemann-Stieltjes integral. In contrast to the usual integral definition known from calculus, the infinite variation of the Wiener process implies that the value of the integral depends on the specific form of the discretization procedure. Here, we demonstrate this on the basis of the most simple example, namely the integral of the Wiener process itself.

Therefore we divide the interval $[t_0, t]$ into n subintervals $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq t$ of equal length, i.e. $t_i = t_0 + i(t - t_0)/n$ and introduce a one-parametric family of stochastic integrals of the Wiener process defined by

$$\int_{t_0}^{t} \hat{W}(t') \overset{\lambda}{\circ} \mathrm{d}\hat{W}(t') := \sum_{i=1}^{n} \left[\lambda \hat{W}(t_i) + (1-\lambda)\hat{W}(t_{i-1})\right] \left[\hat{W}(t_i) - \hat{W}(t_{i-1})\right] . \tag{3}$$

The case $\lambda = 0$ corresponds to the so-called Itô integral, while $\lambda = 1/2$ yields the Stratonovich definition of the stochastic integral.

2.1. Expectation value of the generalized stochastic integral

Calculate the mean value of the integrals

$$\left\langle \int_{t_0}^{t} \hat{W}(t') \overset{\lambda}{\circ} \mathrm{d}\hat{W}(t') \right\rangle \tag{4}$$

What is the result for $\lambda = 0$, i.e., for the Itô integral. How could one have seen this beforehand?

3. Problem: Correlation formula for the Itô integral (no points)

In general, the Itô integral is defined for an arbitrary non-anticipating function $\hat{g}(t)$, i.e., under the condition that $\hat{g}(t)$ is independent of $\hat{W}(t') - \hat{W}(t)$ for $t < t'$, by

$$\int_{t_0}^{t} \hat{g}(t') \, \mathrm{d}\hat{W}(t') := \sum_{i=1}^{n} \hat{g}(t_{i-1}) \left[\hat{W}(t_i) - \hat{W}(t_{i-1})\right]$$

with the times t_i as given above. Calculate the mean value

$$\left\langle \int\limits_{t_0}^{t} \hat{g}(t')\,\mathrm{d}\hat{W}(t') \int\limits_{t_0}^{t} \hat{h}(t')\,\mathrm{d}\hat{W}(t') \right\rangle .$$

where $\hat{h}(t)$ is another, arbitrary, non-anticipating function.

Compare the above result to the mean value of

$$\left\langle \int\limits_{t_0}^{t} \hat{g}(t')\,\mathrm{d}\hat{W}(t') \left[\int\limits_{t_0}^{t'} \hat{h}(t'')\,\mathrm{d}\hat{W}(t'') \right] \right\rangle .$$

What is the difference between the two expressions?