1. Problem: Branching processes (10 points)

The so-called *branching process* models a chain reaction occuring in successive generations. Initially, for the zeroth generation, we start with a single event. This event causes in the first generation $\hat{r}$ further events, where $\hat{r}$ is a random non-negative integer with distribution p_r. If $\hat{r} = 0$, the process stops. Otherwise, every direct descendant causes in the next (second) generation further events according to the same distribution p_r and so on. The events of each generation act independently of each other.

Whereas an obvious example for such a process are nuclear chain reactions, originally this process has been invented by F. Galton to study the survival of family names (hence the name *Galton-Watson process*). Another example is the cascading failure of components like power lines which is induced by an overloading occuring after the initial failure of a line.

1.1. Recursion relation

Let $\hat{y}_n$ be the number of events, i.e., the size of the n-th generation with generating function $f_n(z) = \langle z^{\hat{y}_n} \rangle$. In particular, $y_0 = 1$ in the 0th generation. Furthermore, $f_1(z) = f(z)$, where $f(z) = \langle z^{\hat{r}} \rangle = \sum_{r=0}^{\infty} p_r z^r$ is the generating function corresponding to the variable $\hat{r}$. Derive from the description of the branching process the recursive relation

$$f_n(z) = f(f_{n-1}(z)) \,. \tag{1}$$

1.2. Probability for a finite cascade

An important question is whether the branching process continues forever or stops after a finite number of generations, i.e., whether $\hat{y}_n = 0$ for an arbitrary n.

1.2.1. Recursion formula

How can the probability $q_n := P(\hat{y}_n = 0)$ that the process stops at or before the n-th generation be calculated from the generating function $f_n(z)$? Derive from Eq. (1) a recursion formula for the q_n. What is the initial condition q_1?

1.2.2. Termination of branching process

Excluding the extreme cases $p_0 = 0$ and $p_0 = 1$ derive from the form of the generating function $f(z)$ that the q_n approach a limiting value $q \le 1$ satisfying

$$q = f(q)\,. \tag{2}$$

Show geometrically that equation (2) has the unique solution $q = 1$ if and only if the first derivative $f'(1)$ of the characteristic function satisfies $f'(1) \le 1$.
Hint: The characteristic function is convex. Why?

What does condition (2) yield for the special case of the variable $\hat{r}$ being a Poisson distribution with mean a?

1.2.3. Mean size of generations

Show that the expected size of the n-th generation is given by

$$\langle \hat{y}_n \rangle = f_n'(1).$$

Derive a chain rule for $\langle \hat{y}_n \rangle$ and show that

$$\langle \hat{y}_n \rangle = \mu^n,$$

where μ is the mean value of $\hat{r}$. Why is it intuitively clear that the branching process dies out for $f'(1) < 1$?

2. Problem: Compound distribution (no points)

Let $\hat{x}_j$ be an infinite set of independent stochastic variables with identical distributions $P_{\hat{x}}(x)$ and characteristic function $G_{\hat{x}}(k) = \int_{-\infty}^{\infty} P_{\hat{x}}(x) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x$. Let $\hat{r}$ be a random non-negative integer with distribution p_r and probability generating function $f(z) = \langle z^{\hat{r}} \rangle = \sum_{r=0}^{\infty} p_r z^r$. Then the sum

$$\hat{y} = \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\hat{r}}$$

is a random variable, where we set $\hat{y} = 0$ for $\hat{r} = 0$. Show that its characteristic function $G_{\hat{y}}(k)$ fulfills

$$G_{\hat{y}}(k) = f(G_{\hat{x}}(k)).$$

The distribution of $\hat{y}$ is the so-called *compound distribution.*

2.1. Example

Assume that $\hat{x}_i$ are Bernoulli variables, i.e. $P(\hat{x}_i = 1) = p = 1 - q$ and $P(\hat{x}_i = -1) = q$ like in the asymetric random walk discussed on problem set 2 and $\hat{r}$ obeys Poisson statistics. How does the compound distribution $P_{\hat{y}}$ then look like?

3. Problem: Box-Muller algorithm for the generation of Gaussian random variables (no points)

For the generation of random numbers according to a certain distribution on a computer, one usually starts with uniformly distributed pseudo-random numbers[1] and then does a suitable transformation to the desired random variable. One of the most famous methods, the so-called Box-Muller transform, deals with the generation of Gaussian random numbers.

Show that from two independent uniformly distributed in the interval $(0, 1]$ random numbers $\hat{x}_1$ and $\hat{x}_2$, one obtains two independent Gaussian variables $\hat{z}_1$ and $\hat{z}_2$ with zero mean and unit variance by means of the transformation

$$\hat{z}_1 = \sqrt{-2 \ln \hat{x}_1} \, \cos(2\pi \hat{x}_2) \tag{3}$$

$$\hat{z}_2 = \sqrt{-2 \ln \hat{x}_1} \, \sin(2\pi \hat{x}_2) \tag{4}$$

Hint: Write the joint probability density of the transformed variables using a two-dimensional δ-function and then evaluate this expression using the Jacobian $\frac{\partial(x_1,x_2)}{\partial(z_1,z_2)}$ of the transformation.

[1] Pseudo-random number generators are a whole topic by itself, which we do not cover here. For an excellent overview see, e.g., Chapter 3 in Donald E. Knuth, The Art of Computer Programming, Volume 2: Seminumerical Algorithms, 3rd edition (Addison-Wesley, 1997)