1. Problem: Shot noise

1.1. Introduction

Consider the electrical current $\hat{I}(t)$ induced by a fixed voltage across a resistor. Resulting from the motion of single charge carriers, this current will fluctuate in time around its mean $\langle \hat{I}(t) \rangle$, an effect we model by describing it as a random process. For the moment, we only assume that this process is stationary, which implies that the mean is in fact time-independent: $\langle \hat{I}(t) \rangle =: I$.

In the following, we will derive a relation for the strength of the current fluctuations, the so-called current noise Doing so, we will use a trick, namely, we will not consider the current itself but its time-integral, i.e. the charge.

The reason for this is that the current $\hat{I}(t)$ itself is in fact a not very well-behaved process: every realization consists of a sum of narrow peaks—one for every charge transfer event—with a width determined by properties of the charge detector. In an idealized picture, we would approximate these peaks by δ-functions, and the current would consist of a sum of such δ-peaks at random instances in time. When calculating the noise, which is related to correlation functions, we obtain products of such δ-functions, which leads to several problem, which we will circumvent with a charge-based approach.

1.2. Charge-number process

Let us thus look at the electric charge $e\hat{N}(t)$ transfered through the resistor between time 0 and time t:

$$e\hat{N}(t) := \int_0^t \mathrm{d}t' \, \hat{I}(t'), \tag{1}$$

Is the process $\hat{N}(t)$ counting the number of transfered charges stationary, as well?

So far, we have not made any assumption concerning the current $\hat{I}(t)$, beside its stationarity. We will now do so implicitely by assuming that the transferred charge $\hat{N}(t)$ is described by a

Poisson process with parameter λ:

$$\mathrm{Prob}\big(\hat{N}(t) = N\big) = p_N(t) = \mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^N}{N!}. \tag{2}$$

Check that this assumption is consistent with the definition (1) at time $t = 0$.

Furthermore, what does it imply for the sign of the current $\hat{I}(t)$? What physical situation (temperature, voltage) do we thus restrict ourselves?

Sketch a couple of realizations of the process $\hat{N}(t)$ and the corresponding current $\hat{I}(t)$.

To determine the parameter λ, express it in terms the average current I.

Hint: Take expectation values of both sides of Eq. (1)

1.3. MacDonald's theorem

We will now the derived, the desired relation for the current noise, which is defined as the Fourier transform

$$S_{\hat{I}}(\omega) := \int \mathrm{d}\tau\, \mathrm{e}^{i\omega\tau} S_{\hat{I}}(\tau),$$

of the current autocorrelation function $S_{\hat{I}}(\tau) := \langle\langle \hat{I}(\tau)\hat{I}(0)\rangle\rangle = \langle \big[\hat{I}(\tau) - I\big]\big[\hat{I}(0) - I\big]\rangle$. Thus, $S_{\hat{I}}(\omega)$ is the spectral density of the current noise $\hat{I}(t)$.

Proof MacDonald's theorem

$$S_{\hat{I}}(\omega) = \omega \int_0^\infty \mathrm{d}t\, \sin(\omega t)\, \frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \hat{N}(t)^2\rangle\rangle \tag{3}$$

which relates this noise with the fluctuations of the time-integral. The upper limit of the above equation is guaranteed by a converging factor $\mathrm{e}^{-\varepsilon t}$.

Note that this proof will be very general, i.e., it does not rely on the previous assumption that $\hat{N}(t)$ is a Poisson process. It requires that $\hat{I}(t)$ is stationary and real, though. Why can this already been seen from (3)?

Hint: To proof Eq. (3) first derive the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \hat{N}(t)^2\rangle\rangle = 2\int_0^t \mathrm{d}\tau\; S_{\hat{I}}(\tau)$$

Considering again the special case of $\hat{N}(t)$ being the Poisson process (2), derive from the McDonald theorem the so-called Schottky's theorem

$$S_{\hat{I}}(\omega) = eI.$$

Often, this is expressed in terms of the Fano factor $F = S_{\hat{I}}(\omega)/e\langle I \rangle = 1$.

2. Problem: Sinusoidal process with random amplitude and phase (no points)

Consider a stochastic process $\hat{x}(t) = \hat{A}\sin(\omega t + \hat{\phi})$ with fixed (deterministic) angular frequency ω but random real amplitude $\hat{A}$ and phase $\hat{\phi}$, which we assume to be independent but otherwise arbitrary except that the second moment $\langle \hat{A}^2 \rangle$ has to be finite.

2.1. Mean and autocorrelation function

Express the mean $\langle \hat{x}(t) \rangle$ and the autocorrelation function $S(t_1, t_2) = \langle [\hat{x}(t_1) - \langle \hat{x}(t_1) \rangle] [\hat{x}(t_2) - \langle \hat{x}(t_2) \rangle] \rangle$ of the process in terms of the moments of $\hat{A}$ and the characteristic function of $\hat{\phi}$.

What *necessary* condition does $\hat{\phi}$ have to fulfill in order for the process $\hat{x}(t)$ to be stationary, i.e., under which conditions is $\langle \hat{x}(t) \rangle = \text{const.}$ and $S(t_1, t_2) = S(t_1 - t_2, 0)$.

2.2. Uniformly distributed phase

Consider now the special case that the phase is uniformely distributed on the interval $[0, 2\pi]$.

Calculate the mean and autocorrelation in this case.

Show that the process is now (strictly) stationary, i.e., that $\hat{x}(t)$ and $\hat{x}(t+\tau)$ have the same same hierarchy of distribution functions.

Hint: Consider the process $\hat{\psi} = (\omega\tau + \hat{\phi}) \bmod 2\pi$.

3. Problem: Correlation functions and the Cauchy-Schwarz inequality (no points)

3.1. Semi-positivity of the autocorrelation function

Show that for arbitrary times $t_1, \ldots, t_N$ and complex numbers $a_1, \ldots a_N$ the following relation for the autocorrelation function of an arbitrary stochastic process $\hat{x}(t)$ holds true:

$$\sum_{i,j=1}^{N} a_i^* \, S(t_i, t_j) \, a_j \geq 0 \,. \tag{4}$$

Here, z^* denotes the complex conjugate of the complex number z.

Hint: Consider the expectation value $\big\langle \big| \sum_{i=1}^{N} a_i^* \, \delta\hat{x}(t_i) \big|^2 \big\rangle$, where $\delta\hat{x}(t) = \hat{x}(t) - \langle \hat{x}(t) \rangle$ are the fluctuations of the process $\hat{x}(t)$.

3.2. Cauchy-Schwarz inequality

Write the identity (4) for the special case $N = 2$ in the form

$$\mathbf{a}^\dagger \, \mathsf{M}(t_1, t_2) \, \mathbf{a} \geq 0 \quad \text{for } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{and a } 2 \times 2 \text{ hermitian matrix } \mathsf{M}(t_1, t_2). \tag{5}$$

In this form, the identity means that the matrix $\mathsf{M}(t_1, t_2)$ is positive semi-definite.

Deduce from the fact that the determinant of this matrix has to be non-negative (why?) the Cauchy-Schwarz inequality

$$\big\langle \big| \delta\hat{x}(t_1) \big|^2 \big\rangle \big\langle \big| \delta\hat{x}(t_2) \big|^2 \big\rangle \geq \big| \big\langle \delta\hat{x}(t_1) \, \delta\hat{x}(t_2)^* \big\rangle \big|^2 \tag{6}$$