1. Problem: Fokker-Planck equation (0 points)

We consider a Markov process whose conditional probability $p_{1|1}(x,t|x_0,t_0)$ is governed by the Fokker-Planck equation

$$\frac{\partial p_{1|1}(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}\left[A(x,t)\,p_{1|1}(x,t|x_0,t_0)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t)\,p_{1|1}(x,t|x_0,t_0)\right] \tag{1}$$

with the initial condition $p_{1|1}(x,t_0|x_0,t_0) = \delta(x-x_0)$ as required by the definition of the conditional probability. The coefficient $A(x,t)$ can be arbitrary but we require $D(x,t) > 0$.

Convince yourself that the one-time probability $p_1(x,t)$ of the process also has to fulfil an equation of the form (1). What is the initial condition?

Assume $t = t_0 + \Delta t$ in the Fokker-Planck equation for $p_{1|1}(x,t|x_0,t_0)$. Compute for small Δt the moments of $\Delta x = x - x_0$. Show that for $\Delta t \to 0$ the conditional expectation values obey

$$\begin{aligned} \frac{\langle \Delta x|x_0,t_0\rangle}{\Delta t} &= A(x_0,t_0), \\ \frac{\langle (\Delta x)^2|x_0,t_0\rangle}{\Delta t} &= 2D(x_0,t_0), \\ \frac{\langle (\Delta x)^\nu|x_0,t_0\rangle}{\Delta t} &= 0 \quad \text{for} \quad \nu \geq 3. \end{aligned}$$

Interpret this result. What is the descriptive meaning of the two coefficients A and D?

2. Problem: No perpetuum mobile of the second kind (10 points)

We now consider a process that takes place on an interval $[a,b]$ in which the two end points are identified with each other, e.g. diffusion in presence of an external force (drift) on a circle. A promiment example is that of a ratchet and a pawl discussed in the Feynman Lectures on Physics (Vol. I, Chapter 46), where x can be identified with the angle of rotation of the ratchet.

It is useful to rewrite the Fokker-Planck equation (1) in the form of a conservation law for the probability:

$$\frac{\partial p(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0\,. \tag{2}$$

Here, we have introduced the probability current

$$j(x,t) = A(x,t)\,p(x,t) - \frac{\partial}{\partial x}\left[D(x,t)\,p(x,t)\right] . \tag{3}$$

Note that since both $p_1(x,t)$ and $p_{1|1}(x,t|x_0,t_0)$ obey the same dynamics, and only differ in their initial conditions, we use here and in the following the more compact, but also a bit sloppy, notation $p(x,t)$.

Argue that for the periodic boundary conditions discussed above, we have to require the matching conditions:

$$\lim_{x\to b^-} p(x,t) = \lim_{x\to a^+} p(x,t) \tag{4}$$

$$\lim_{x\to b^-} j(x,t) = \lim_{x\to a^+} j(x,t) . \tag{5}$$

2.1. Stationary solution for time-homogeneous process

Now we consider the stationary solution $p(x,t) = p_\mathrm{s}(x)$ assuming a time-homeogeneous process such that the drift and diffusion constant are time independent: $A(x,t) = A(x)$ and $D(x,t) = D(x)$.

Show that the stationary current $j(x,t) = j_\mathrm{s}(x)$ then has to be x-independent, as well:

$$j_\mathrm{s}(x) = \text{const.} = j_\mathrm{s}.$$

2.2. Current for periodic boundary conditions

In contrast to the situation with, e.g., reflecting boundary conditions, where one has to require $j_\mathrm{s} = 0$, the current does not need to vanish in the presence of periodic boundary conditions. It is determined by the normalization condition for the probability density and the periodic boundary conditions (4, 5). Integrate Eq. (3) to find

$$j_\mathrm{s} = p_\mathrm{s}(a)\left[\frac{D(a)}{\Psi(a)} - \frac{D(b)}{\Psi(b)}\right]\left[\int_a^b \frac{\mathrm{d}x'}{\Psi(x')}\right]^{-1} , \tag{6}$$

where we have introduced the function

$$\Psi(x) = \exp\left[\int_a^x \mathrm{d}x'\,\frac{A(x')}{D(x')}\right] .$$

Derive an expression for the stationary probability $p_\mathrm{s}(x)$ from this expression.

2.3. Condition for vanishing current

Assume now that $D(x) = D$ with the position independent diffusion constant D. The sign of the current depends on the sign of the prefactor $[1/\Psi(b) - 1/\Psi(a)]$ as derived in the previous section. What is the condition that the current j_s is zero, positive or negative?

Give a simple sufficient condition on the potential of $A(x)$ which guarantees that $j = 0$. Interprete this hopefully surprising result—think about an asymmetric, "ratchet", potential—in terms of the second law of thermodynamics.

In order to generate a ratchet current even in the absence of a net force, i.e., to obtain work from fluctuations, one has to consider non-equilibrium situations, e.g. by introducing a time-periodic external force or non-equilibrium noise. A plethora of such Brownian motor models has been discussed in the literature, see e.g. P. Reimann, *Brownian motors: noisy transport far from equilibrium*, Phys. Rep. **361**, 57 (2002) for a comprehensive review.