



1. Problem: Shot noise (10 points)

1.1. Introduction

Consider the electrical current $\hat{I}(t)$ induced by a fixed voltage across a resistor. Resulting from the motion of single charge carriers, this current will fluctuate in time around its mean $\langle \hat{I}(t) \rangle$, an effect we model by describing it as a random process. For the moment, we only assume that this process is stationary, which implies that the mean is in fact time-independent: $\langle \hat{I}(t) \rangle =: I$.

In the following, we will derive a relation for the strength of the current fluctuations, the so-called current noise. As a trick, we will not consider the current itself but its time-integral, i.e., the charge.

The reason for this is that the current $\hat{I}(t)$ itself is in fact a not very well-behaved process: every realization consists of a sum of narrow peaks—one for every charge transfer event—with a width determined by properties of the charge detector. In an idealized picture, we would approximate these peaks by δ -functions, and the current would consist of a sum of such δ -peaks at random instances in time. When calculating the noise, which is related to correlation functions, we obtain products of such δ -functions, which leads to several problems, which we will circumvent with a charge-based approach.

1.2. Charge-number process

Let us thus look at the electric charge $e\hat{N}(t)$ transferred through the resistor between time 0 and time t :

$$e\hat{N}(t) := \int_0^t dt' \hat{I}(t'), \quad (1)$$

Is the process $\hat{N}(t)$ counting the number of transferred charges stationary, as well?

So far, we have not made any assumption concerning the current $\hat{I}(t)$, beside its stationarity. We will now do so implicitly by assuming that the transferred charge $\hat{N}(t)$ is described by a

Poisson process with parameter λ :

$$\text{Prob}(\hat{N}(t) = N) = p_N(t) = e^{-\lambda t} \frac{(\lambda t)^N}{N!}. \quad (2)$$

Check that this assumption is consistent with the definition (1) at time $t = 0$.

Furthermore, what does it imply for the sign of the current $\hat{I}(t)$? What physical situation (temperature, voltage) do we thus restrict ourselves?

Sketch a couple of realizations of the process $\hat{N}(t)$ and the corresponding current $\hat{I}(t)$.

To determine the parameter λ , express it in terms the average current I .

Hint: Take expectation values of both sides of Eq. (1).

1.3. MacDonald's theorem

We will now derive, the desired relation for the current noise, which is defined as the Fourier transform

$$S_{\hat{I}}(\omega) := \int d\tau e^{i\omega\tau} S_{\hat{I}}(\tau),$$

of the current autocorrelation function $S_{\hat{I}}(\tau) := \langle\langle \hat{I}(\tau)\hat{I}(0) \rangle\rangle = \langle[\hat{I}(\tau) - I][\hat{I}(0) - I]\rangle$. Thus, $S_{\hat{I}}(\omega)$ is the spectral density of the current noise $\hat{I}(t)$.

Proof MacDonald's theorem

$$S_{\hat{I}}(\omega) = \omega \int_0^{\infty} dt \sin(\omega t) \frac{d}{dt} \langle\langle \hat{N}(t)^2 \rangle\rangle \quad (3)$$

which relates this noise with the fluctuations of the time-integral. The upper limit of the above equation is guaranteed by a converging factor $e^{-\epsilon t}$.

Note that this proof will be very general, i.e., it does not rely on the previous assumption that $\hat{N}(t)$ is a Poisson process. It requires that $\hat{I}(t)$ is stationary and real, though. Why can this already be seen from (3)?

Hint: To proof Eq. (3) first derive the relation

$$\frac{d}{dt} \langle\langle \hat{N}(t)^2 \rangle\rangle = 2 \int_0^t d\tau S_{\hat{I}}(\tau)$$

Considering again the special case of $\hat{N}(t)$ being the Poisson process (2), derive from the McDonald theorem the so-called Schottky's theorem

$$S_{\hat{I}}(\omega) = eI.$$

Often, this is expressed in terms of the Fano factor $F = S_{\hat{I}}(\omega)/e\langle I \rangle = 1$.

2. Problem: Sinusoidal process with random amplitude and phase (no points)

Consider a stochastic process $\hat{x}(t) = \hat{A} \sin(\omega t + \hat{\phi})$ with fixed (deterministic) angular frequency ω but random real amplitude \hat{A} and phase $\hat{\phi}$, which we assume to be independent but otherwise arbitrary except that the second moment $\langle \hat{A}^2 \rangle$ has to be finite.

2.1. Mean and autocorrelation function

Express the mean $\langle \hat{x}(t) \rangle$ and the autocorrelation function $S(t_1, t_2) = \langle [\hat{x}(t_1) - \langle \hat{x}(t_1) \rangle] [\hat{x}(t_2) - \langle \hat{x}(t_2) \rangle] \rangle$ of the process in terms of the moments of \hat{A} and the characteristic function of $\hat{\phi}$.

What *necessary* condition does $\hat{\phi}$ have to fulfill in order for the process $\hat{x}(t)$ to be stationary, i.e., under which conditions is $\langle \hat{x}(t) \rangle = \text{const.}$ and $S(t_1, t_2) = S(t_1 - t_2, 0)$.

2.2. Uniformly distributed phase

Consider now the special case that the phase is uniformly distributed on the interval $[0, 2\pi]$.

Calculate the mean and autocorrelation in this case.

Show that the process is now (strictly) stationary, i.e., that $\hat{x}(t)$ and $\hat{x}(t + \tau)$ have the same hierarchy of distribution functions.

Hint: Consider the process $\hat{\psi} = (\omega\tau + \hat{\phi}) \bmod 2\pi$.

3. Problem: Correlation functions and the Cauchy-Schwarz inequality (no points)

3.1. Semi-positivity of the autocorrelation function

Show that for arbitrary times t_1, \dots, t_N and complex numbers a_1, \dots, a_N the following relation for the autocorrelation function of an arbitrary stochastic process $\hat{x}(t)$ holds true:

$$\sum_{i,j=1}^N a_i^* S(t_i, t_j) a_j \geq 0. \quad (4)$$

Here, z^* denotes the complex conjugate of the complex number z .

Hint: Consider the expectation value $\langle |\sum_{i=1}^N a_i^* \delta\hat{x}(t_i)|^2 \rangle$, where $\delta\hat{x}(t) = \hat{x}(t) - \langle \hat{x}(t) \rangle$ are the fluctuations of the process $\hat{x}(t)$.

3.2. Cauchy-Schwarz inequality

Write the identity (4) for the special case $N = 2$ in the form

$$\mathbf{a}^\dagger \mathbf{M}(t_1, t_2) \mathbf{a} \geq 0 \quad \text{for } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and a } 2 \times 2 \text{ hermitian matrix } \mathbf{M}(t_1, t_2). \quad (5)$$

In this form, the identity means that the matrix $\mathbf{M}(t_1, t_2)$ is positive semi-definite.

Deduce from the fact that the determinant of this matrix has to be non-negative (why?) the Cauchy-Schwarz inequality

$$\langle |\delta\hat{x}(t_1)|^2 \rangle \langle |\delta\hat{x}(t_2)|^2 \rangle \geq |\langle \delta\hat{x}(t_1) \delta\hat{x}(t_2)^* \rangle|^2 \quad (6)$$